

An introduction to strict quantization

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Abstract

We present a short review of the approach to quantization known as *strict (deformation) quantization*, which can be seen as a generalization of the Weyl-Moyal quantization. We include examples and comments on the process of quantization.

1 Introduction

This brief review is my modest contribution to a field and a set of ideias that strongly influenced my view on the process of *quantization*. I hope that this introduction to the subject of *strict quantization* and the examples in it can be of help, and motivate both theoretical and mathematical physicists to this beautiful and highly developed field. My personal impression is that there is still a whole lot to explore in this area, in particular concerning extensions of the formalism to the realm of infinite dimensional degrees of freedom. In fact, it seems clear that quantum field theory could benefit greatly from a strict quantization approach. Concrete steps have been taken in this direction (see e.g. [1, 2, 3] and also [4] for a recent application in quantum gravity), in what is for sure a rather promising area in mathematical physics.

This brief survey focus on finite dimensional examples exclusively, and is mostly inspired by (parts of) two beautiful books, namely Folland's *Harmonic Analysis on Phase Space* [5] and Landsman's synthesis *Mathematical Topics Between Classical and Quantum Mechanics* [6]. Another major source of inspiration was Várilly's book [7]. Concerning broader fields, such as Poisson geometry, groupoids or deformation of algebra structures, we follow also [8, 9], as well as [10]. An effort is made, hopefully sucessful, to keep the discussion short and as pedagogical as possible, just enough to give the reader a grasp of what strict quantization is all about and to motivate him/her to this field.

This work is organized as follows. In the remaining of the present section we give a very brief introduction to the problem of quantization. In section 2 we review the formalism of classical mechanics and prepare for the strict quantization approach. In section 3 we review the Weyl-Moyal quantization, which is the prototype of a strict deformation quantization. In section 4 we present what is essentially Landsman's definition of strict quantization. In the following sections we present methods for the construction of strict quantizations which, in one way or another, generalize the Weyl-Moyal quantization process. First, in section 5 we review the notions of smooth groupoids and associated convolution algebras. Then Connes' tangent groupoid and Landsman's strict quantization of cotangent bundles of Riemannian manifolds are discussed, in section 6. Finally, we discuss the interesting case

of the 2-torus, treated both by geometric quantization related and by strict quantization methods.

The formalism of classical mechanics is based on a phase space (the states) and on functions on that space (the observables), on the set of which two operations are defined: the product of functions and the Poisson bracket. Quantum mechanics presents a similar structure, keeping the duality states-observables. In fact, in quantum mechanics one deals with operators (observables) in a Hilbert space (the states). The operation that combines observables is now the operator product. One can extend further the analogy with the classical structure, by means of two new operations, obtained by symmetrization and antisymmetrization of the operator product. The anticommutator \circ , or Jordan product, is given by

$$A \circ B := \frac{1}{2}(AB + BA) . \quad (1)$$

Following Dirac [11], one defines also the quantum Lie bracket, or quantum Poisson bracket $[\cdot, \cdot]_{\hbar}$ by

$$[A, B]_{\hbar} := (AB - BA)/i\hbar , \quad (2)$$

where $\hbar = h/2\pi$, with h being Planck's constant. Of course, we have

$$AB = A \circ B + \frac{i\hbar}{2}[A, B]_{\hbar} . \quad (3)$$

It is tempting to interpret the Jordan product as the quantum equivalent of the product of functions, together with Dirac's quantum condition stating that the quantum bracket is to be seen as the quantum correspondent of the Poisson bracket. In broad terms, the transition from a classical description to a quantum description of a given system - *quantization* - is obtained by means of a map \mathcal{Q} from functions on phase space to operators on a Hilbert space. In this process, one aims at preserving the relations between observables as much as possible. It would therefore be natural to search for maps \mathcal{Q} such that

$$\mathcal{Q}(fg) = \mathcal{Q}(f) \circ \mathcal{Q}(g) , \quad (4)$$

$$\mathcal{Q}(\{f, g\}) = [\mathcal{Q}(f), \mathcal{Q}(g)]_{\hbar} . \quad (5)$$

However, it is well known that such maps cannot be found. Even Dirac's condition (5) (supplemented with natural requirements, see below) is impossible to fulfil exactly, except for a restricted set of observables (and typically on cotangent spaces only). Conditions (4) and (5) should therefore be replaced by weaker ones, still keeping the purpose of rigorously implementing the physical requirement embodied by those relations. The straightforward *canonical quantization* approach, launched by Dirac, consists in totally relaxing condition (4) and implementing condition (5) exactly on a small, but sufficiently large, subalgebra of observables. Further crucial requirements must be added in this approach, such as irreducibility (see e.g. [12] for a general discussion of canonical quantization).

The *strict quantization* approach [6, 13] proposes an asymptotic implementation of conditions (4) and (5). It involves not a single map \mathcal{Q} , but a family of maps, labeled by a parameter which we will call \hbar , although this is now a free parameter and no longer the value of the physical constant. One then considers a family of maps \mathcal{Q}_{\hbar} , satisfying relations of the type

$$\lim_{\hbar \rightarrow 0} \left(\mathcal{Q}_{\hbar}(fg) - \mathcal{Q}_{\hbar}(f) \circ \mathcal{Q}_{\hbar}(g) \right) = 0 , \quad (6)$$

$$\lim_{\hbar \rightarrow 0} \left(\mathcal{Q}_\hbar(\{f, g\}) - [\mathcal{Q}_\hbar(f), \mathcal{Q}_\hbar(g)]_\hbar \right) = 0 . \quad (7)$$

Let us admit further that each of the \mathcal{Q}_\hbar is injective and that, as a function of \hbar , \mathcal{Q}_\hbar and \mathcal{Q}_\hbar^{-1} are continuous. In this case, the operator product induces a family of associative operations on the functions in phase space:

$$f *_\hbar g := \mathcal{Q}_\hbar^{-1} \left(\mathcal{Q}_\hbar(f) \mathcal{Q}_\hbar(g) \right) , \quad (8)$$

which is a deformation of the standard product, in the sense that

$$\lim_{\hbar \rightarrow 0} f *_\hbar g = fg , \quad \lim_{\hbar \rightarrow 0} (f *_\hbar g - g *_\hbar f) / i\hbar = \{f, g\} . \quad (9)$$

2 Classical systems

Given a C^∞ manifold \mathcal{P} , let $C^\infty(\mathcal{P})$ denote the associative algebra of C^∞ complex functions on \mathcal{P} , equipped with the usual product and involution given by complex conjugation, $f \mapsto \bar{f}$.

Definition 1 *A Poisson structure on a manifold \mathcal{P} is a bilinear operation $\{, \}$ on $C^\infty(\mathcal{P})$ such that:*

- (i) $(C^\infty(\mathcal{P}), \{, \})$ is a Lie algebra.
- (ii) $\{f, \cdot\}$ is a derivation on $C^\infty(\mathcal{P})$ for every $f \in C^\infty(\mathcal{P})$.
- (iii) $\{\bar{f}, \bar{g}\} = \overline{\{f, g\}}$.

A manifold with a Poisson structure is said to be a Poisson manifold.

Given local coordinates (x_1, \dots, x_n) in \mathcal{P} , the Poisson bracket is given by

$$\{f, g\} = \Pi^{ij}(x) \partial_i f \partial_j g , \quad (10)$$

where $\partial_i f := \frac{\partial f}{\partial x_i}$ and the real quantities $\Pi^{ij}(x)$ constitute the components of a contravariant antisymmetric (real) 2-tensor

$$\Pi = \Pi^{ij}(x) \partial_i \otimes \partial_j . \quad (11)$$

The latter is said to be a Poisson tensor and satisfies the condition

$$\Pi^{li} \partial_l \Pi^{jk} + \Pi^{lj} \partial_l \Pi^{ki} + \Pi^{lk} \partial_l \Pi^{ij} = 0 . \quad (12)$$

Conversely, an antisymmetric (real) tensor Π that fulfils (12) defines a Poisson structure $\{, \}$ by

$$\{f, g\} = \Pi(df, dg) . \quad (13)$$

Straightforward examples of Poisson manifolds are provided by symplectic manifolds, e.g. pairs (\mathcal{P}, ω) where ω is a closed nondegenerate 2-form. The Poisson tensor is in this case also nondegenerate. It is given by the inverse of the symplectic form, or explicitly by

$$\Pi(i_X(\omega), i_Y(\omega)) = \omega(X, Y) , \quad \forall X, Y \in \mathcal{X}(\mathcal{P}) , \quad (14)$$

where $\mathcal{X}(\mathcal{P})$ denotes the space of vector fields on \mathcal{P} . In fact, the property $d\omega = 0$ guarantees that Π (14) satisfies the condition (12).

Poisson manifolds $(\mathcal{P}, \{, \})$ are precisely the mathematical structures taken as models for finite dimensional classical systems, both in classical mechanics and in classical statistical mechanics. The physical interpretation of the formalism stands on the notions of *physical state* and *physical observable* and on the relations between them. In the definition of states and observables below we follow from the start an approach adapted to the C^* -algebras formalism.

Definition 2 *A physical state on a Poisson manifold $(\mathcal{P}, \{, \})$ is a regular Borel probability measure on \mathcal{P} . The atomic measures, identified with points in \mathcal{P} , are said to be pure states, whereas the remaining ones are called mixed states.*

Concerning observables, one typically considers the set of all real C^∞ functions on \mathcal{P} . In this respect let us note the following.

There is no inconvenience in considering complex functions, given that real functions are recovered as the invariant subset under involution. Likewise, when considering the problem of quantization we will work with complex algebras, taking into account the so-called *reality conditions*. So, we will require that real functions belonging to the classical algebra are mapped under quantization to self-adjoint elements of the quantum algebra, which is the same as requiring that the process of quantization maps the involution of functions to the natural involution of operators.

Aiming at the introduction of the C^* -algebra formalism, it is convenient to work with a subspace of $C^\infty(\mathcal{P})$ which is also contained on the C^* -algebra $C_0(\mathcal{P})$ of continuous functions vanishing at infinity¹. If \mathcal{P} is compact the behaviour at infinity is not a question and one can take the full space $C^\infty(\mathcal{P})$. In case \mathcal{P} is only locally compact, several choices appear possible, for instance $C_0^\infty(\mathcal{P})$, the set of C^∞ functions vanishing at infinity, or $C_c^\infty(\mathcal{P})$, the set of C^∞ functions with compact support.

Without claiming to establish what should be meant by *physical observable*, let us adopt the following definition.

Definition 3 *A complete algebra of regular observables on a Poisson manifold $(\mathcal{P}, \{, \})$ is a subspace $\mathcal{A}(\mathcal{P}) \subset C_0^\infty(\mathcal{P})$ such that:*

- (i) $(\mathcal{A}(\mathcal{P}), \{, \})$ is a Lie subalgebra of $(C^\infty(\mathcal{P}), \{, \})$.
- (ii) $\mathcal{A}(\mathcal{P})$ is a dense $*$ -subalgebra (with respect to the supremum norm) of $C_0(\mathcal{P})$.

As examples of such complete algebras one can mention $C_0^\infty(\mathcal{P})$ and $C_c^\infty(\mathcal{P})$, which are well defined in any circumstance. Nonetheless, other choices may be more convenient, in a given particular situation. In any case, we will consider chosen a complete algebra of regular observables $\mathcal{A}(\mathcal{P})$ such that $C_c^\infty(\mathcal{P}) \subseteq \mathcal{A}(\mathcal{P}) \subseteq C_0^\infty(\mathcal{P})$.

Such algebras are complete in the following sense: condition (ii) in definition 3 guarantees that $\mathcal{A}(\mathcal{P})$ separates points in \mathcal{P} , i.e. given $x \neq y$ em \mathcal{P} , there exists $f \in \mathcal{A}(\mathcal{P})$ such that $f(x) \neq f(y)$.

¹For locally compact X , $C_0(X)$ is the subset of those $f \in C(X)$ with the property that for any $\epsilon > 0$, there is a compact set $K_\epsilon \subset X$ such that $|f(x)| < \epsilon$ if $x \notin K_\epsilon$. The C^* -norm of $C_0(X)$ is the supremum norm.

Let us clarify immediately the following. Although there are enough functions in a complete algebra of regular observables to define local coordinates, by no means such an algebra contains all observables of physical interest, if \mathcal{P} is noncompact. The obvious example is provided by the standard global coordinate functions in $\mathcal{P} = \mathbb{T}^*\mathbb{R}$. Turning to quantization, the strategy will be to start with a convenient algebra of regular observables, seeking afterwards to extend the quantization map to other observables of interest, if necessary.

The quantities with direct physical correspondence are the (real) values of the pairing

$$(\mu, f) \mapsto \int d\mu(x)f(x) , \quad (15)$$

for real functions f and states μ . Therefore, it is usually said that the description of the system in terms of states is “dual” to the description in terms of observables. In the context of definitions 2 and 3, this duality has a precise sense: the physical states show up as a subset of the unit ball in the dual of $\mathcal{A}(\mathcal{P})$ (let us remind that $\mathcal{A}(\mathcal{P})$ is dense in $C_0(\mathcal{P})$, and therefore the dual of $\mathcal{A}(\mathcal{P})$ coincides with the dual of $C_0(\mathcal{P})$). To be precise, let us introduce the following:

Definition 4 *A linear functional φ on a $*$ -algebra \mathcal{A} is said to be positive if $\varphi(a^*a) \geq 0 \ \forall a \in \mathcal{A}$ (we write $\varphi \geq 0$ to denote that φ is positive). A positive linear functional on a C^* -algebra is called a state (of the algebra) if $\|\varphi\| = 1$.*

Note that a positive linear functional on a C^* -algebra is necessarily continuous (see e.g. [14]).

Proposition 1 *Let X be a locally compact Hausdorff space. Then the set of states of the algebra $C_0(X)$ can be identified with the set of regular Borel probability measures on X .*

The proof of this result follows from the Riez-Markov theorem for locally compact spaces (see [15]), which identifies the dual of $C_0(X)$ with the set of finite complex regular Borel measures in X , by means of the bijective correspondence

$$\varphi \mapsto \mu_\varphi : \varphi(f) = \int f d\mu_\varphi , \forall f \in C_0(X). \quad (16)$$

The remaining nontrivial part of the proof consists in a typical functional analysis argument, showing that $\|\varphi\| = \mu_\varphi(X)$, which we will not present here.

The physical states of the system $(\mathcal{P}, \{ , \})$ can therefore be seen as the states of the algebra $C_0(\mathcal{P})$. The pure physical states (atomic measures) admit also important characterizations in terms of the algebra $C_0(\mathcal{P})$.

Proposition 2 *Given a locally compact Hausdorff space X , there is bijective correspondence between the following sets:*

- (i) *The set of atomic measures on X .*
- (ii) *The set of nonnull linear functionals φ on the algebra $C_0(X)$ such that $\varphi(ab) = \varphi(a)\varphi(b) \ \forall a, b \in C_0(X)$.*
- (iii) *The set of irreducible representations of the algebra $C_0(X)$.*

The correspondence between (i) and (ii) is well known and constitutes part of Gelfand's theorem. The correspondence with (iii) is easy to check: given that the algebra $C_0(X)$ is commutative, its irreducible representations have dimension 1, and are therefore C^* -algebras morphisms, $\varphi : C_0(X) \rightarrow \mathbb{C}$, i.e., belong to the set defined by (ii). On the other hand, it is clear that each atomic measure gives rise to such a dimension 1 representation.

One can further show that the above sets defined by (i), (ii) and (iii) are equivalent to the set of *pure states of the algebra $C_0(X)$* , with the following definition.

Definition 5 *A state φ on a C^* -algebra is said to be pure if the conditions $\varphi \geq \chi \geq 0$ can only be fulfilled with $\chi = t\varphi$, $t \in [0, 1]$.*

3 Weyl-Moyal quantization

3.1 Quantization map

Let us consider the phase space $\mathcal{P} = T^*\mathbb{R}$, with local coordinates (q, p) , and its canonical symplectic structure, defined by the form

$$\omega = dq \wedge dp . \quad (17)$$

The associated Poisson tensor is:

$$\Pi = \partial_q \otimes \partial_p - \partial_p \otimes \partial_q , \quad (18)$$

and therefore

$$\{f, g\} = \partial_q f \partial_p g - \partial_p f \partial_q g . \quad (19)$$

We choose as algebra of observables $\mathcal{A}(\mathcal{P})$ the space $\mathcal{S}(T^*\mathbb{R}) \cong \mathcal{S}(\mathbb{R}^2)$ of Schwartz functions on $T^*\mathbb{R} \cong \mathbb{R}^2$. In this context, the Weyl-Moyal (W-M) quantization consists of a family of linear maps \mathcal{Q}_\hbar , $\hbar \in \mathbb{R}^+$, from the Schwartz space to operators in $L^2(\mathbb{R})$. Explicitly:

$$\mathcal{S}(T^*\mathbb{R}) \ni f(q, p) \mapsto \mathcal{Q}_\hbar(f) : \quad (20)$$

$$(\mathcal{Q}_\hbar(f)\psi)(q) = \int \frac{dp}{2\pi\hbar} e^{\frac{ip}{\hbar}(q-q')} f\left(\frac{q+q'}{2}, p\right) \psi(q') dq' , \quad \psi \in L^2(\mathbb{R}). \quad (21)$$

We start by showing that the maps \mathcal{Q}_\hbar have image on the subset of Hilbert-Schmidt operators. The operators $\mathcal{Q}_\hbar(f)$, $f \in \mathcal{S}(T^*\mathbb{R})$, are in fact integral operators, of kernel

$$K_\hbar^f(q, q') = \int \frac{dp}{2\pi\hbar} e^{\frac{ip}{\hbar}(q-q')} f\left(\frac{q+q'}{2}, p\right) , \quad (22)$$

which is clearly well defined and belongs to $\mathcal{S}(\mathbb{R}^2)$. We get

$$\begin{aligned} \int dq dq' |K_\hbar^f(q, q')|^2 &= \int dq dq' \int \frac{dp}{2\pi\hbar} e^{\frac{ip}{\hbar}(q'-q)} \bar{f}\left(\frac{q+q'}{2}, p\right) \\ &\cdot \int \frac{dp'}{2\pi\hbar} e^{\frac{ip'}{\hbar}(q-q')} f\left(\frac{q+q'}{2}, p'\right) . \end{aligned} \quad (23)$$

With the new variables

$$v := \frac{q+q'}{2}, \quad w := \frac{q-q'}{\hbar} \quad (24)$$

it follows that

$$\begin{aligned} \int dq dq' |K_h^f(q, q')|^2 &= \int \frac{dp}{2\pi\hbar} dp' dv \frac{dw}{2\pi} e^{iw(p'-p)} \bar{f}(v, p) f(v, p') \\ &= \frac{1}{2\pi\hbar} \int dv dp |f(v, p)|^2 < \infty, \end{aligned} \quad (25)$$

for every $f \in \mathcal{S}(T^*\mathbb{R})$, which shows that $\mathcal{Q}_h(f)$ is an Hilbert-Schmidt operator.

The smallest C^* -subalgebra of $B(L^2(\mathbb{R}))$ that contains the image of \mathcal{Q}_h coincides with the closure in the uniform topology of the set of Hilbert-Schmidt operators. This is the space $\mathcal{K}(L^2(\mathbb{R}))$ of compact operators. Let us adopt $\mathcal{K}(L^2(\mathbb{R}))$ as common range of the maps \mathcal{Q}_h .

It is straightforward to check that the quantization satisfies the reality conditions

$$\mathcal{Q}_h(\bar{f}) = \mathcal{Q}_h^+(f), \quad \forall \hbar \in \mathbb{R}^+, \quad \forall f \in \mathcal{S}(T^*\mathbb{R}). \quad (26)$$

In fact, it is obvious that the kernel K_h^f (22) satisfies

$$K_h^{\bar{f}}(q, q') = \bar{K}_h^f(q', q), \quad (27)$$

which is equivalent to (26).

One can also show (see [6]) that the W-M quantization (21) satisfies the following conditions:

$$(i) \quad \lim_{\hbar \rightarrow 0} \|\mathcal{Q}_h(\{f, g\}) - [\mathcal{Q}_h(f), \mathcal{Q}_h(g)]_\hbar\| = 0. \quad (28)$$

$$(ii) \quad \lim_{\hbar \rightarrow 0} \|\mathcal{Q}_h(fg) - \mathcal{Q}_h(f) \circ \mathcal{Q}_h(g)\| = 0. \quad (29)$$

$$(iii) \quad \text{The maps } \hbar \mapsto \|\mathcal{Q}_h(f)\| \text{ are continuous in } \mathbb{R}^+, \quad \forall f. \quad (30)$$

$$(iv) \quad \lim_{\hbar \rightarrow 0} \|\mathcal{Q}_h(f)\| = \|f\| (= \sup|f|). \quad (31)$$

Condition (i) is the form in which Dirac's quantization condition is implemented in this formalism: the classical Lie structure is not exactly preserved at the quantum level, but it is violated only by operators that tend to zero with \hbar . Condition (ii) plays the same role with respect to the multiplicative structure, ensuring that the algebraic relations between observables are recovered in the limit $\hbar \rightarrow 0$. Conditions (iii) and (iv) establish the continuity of the process and provide, together with (ii), some control over the spectrum of the quantum operators. In particular, conditions (iii) and (iv) establish precisely the continuity (near $\hbar = 0$) of the spectral radius of the quantum operators.

Let us now clarify the relation between the W-M quantization and the usual canonical quantization of the so-called Heisenberg algebra, i.e. the Lie algebra generated by the coordinate functions q and p . In the Dirac quantization, the observables q and p are mapped to operators \hat{q} and \hat{p} in $L^2(\mathbb{R})$, such that

$$(\hat{q}\psi)(q) = q\psi(q) \quad (32)$$

$$(\hat{p}\psi)(q) = -i\hbar \frac{d\psi}{dq}(q). \quad (33)$$

These operators are unbounded, and therefore cannot be defined for all $\psi \in L^2(\mathbb{R})$. It is standard procedure to restrict attention to the Schwartz subspace $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$, which is dense, belongs to the domain of both \hat{q} and \hat{p} and furthermore remains invariant under

the action of both operators. If the action of $\mathcal{Q}_\hbar(f)$ (21) is restricted to vectors $\psi \in \mathcal{S}(\mathbb{R})$, we see immediately that $\mathcal{Q}_\hbar(f)$ remains well defined for a much larger class of functions in $T^*\mathbb{R}$ (see [5] for a detailed discussion). In particular, $\mathcal{Q}_\hbar(q)$ and $\mathcal{Q}_\hbar(p)$ are well defined in $\mathcal{S}(\mathbb{R})$ and coincide with the operators \hat{q} and \hat{p} above. Thus, the W-M quantization is an extension of the canonical quantization \hat{q} and \hat{p} of coordinate functions, to a large class of observables $f(q, p)$.

3.2 Positivity in the Weyl-Moyal quantization

In this section we address the question of positivity in the Weyl-Moyal quantization.

Let us recall that a self-adjoint operator A is said to be positive if its expectation values are nonnegative, i.e. if $\langle \psi, A\psi \rangle \geq 0$, $\forall \psi$. An equivalent condition is that the spectrum of A is a subset of \mathbb{R}_0^+ .

Given the physical interpretation of observables, it is clearly desirable that, under a given quantization, positive classical observables (i.e. those that take only nonnegative values) are mapped to operators which are themselves positive. We show next that such condition is not fully satisfied in the W-M quantization.² However, Heisenberg's uncertainty relation helps in clarifying the situation, showing why a weaker form of positivity is physically acceptable.

For definiteness, let us consider the positive observables given by gaussian functions in phase space, whose quantization is particularly simple. Let then $f_{\alpha, \beta}^{x_0}$ denote the gaussian function:

$$f_{\alpha, \beta}^{x_0}(q, p) = 2 e^{-\frac{1}{2} \frac{(q-q_0)^2}{\alpha}} e^{-\frac{1}{2} \frac{(p-p_0)^2}{\beta}}, \quad (34)$$

with arbitrary $x_0 = (q_0, p_0)$ and $\alpha > 0$, $\beta > 0$. The kernel of the associated operator $\mathcal{Q}_\hbar(f_{\alpha, \beta}^{x_0})$ (21) is easily found to be:

$$K_{\hbar}^{f_{\alpha, \beta}^{x_0}}(q, q') = \bar{\chi}(q) \chi(q') e^{-\frac{1}{8\alpha} (\frac{4\alpha\beta}{\hbar^2} - 1)(q-q')^2}, \quad (35)$$

where χ is an element of $\mathcal{S}(\mathbb{R})$ given by

$$\chi(q) = \left(\frac{2\beta}{\pi\hbar^2} \right)^{1/4} e^{-\frac{1}{4\alpha}(q-q_0)^2} e^{-\frac{ip_0}{\hbar}(q-q_0)}. \quad (36)$$

To address the question of positivity of $\mathcal{Q}_\hbar(f_{\alpha, \beta}^{x_0})$ let us then consider the expectation values $\langle \psi, \mathcal{Q}_\hbar(f_{\alpha, \beta}^{x_0})\psi \rangle$, $\psi \in L^2(\mathbb{R})$. We get

$$\begin{aligned} \langle \psi, \mathcal{Q}_\hbar(f_{\alpha, \beta}^{x_0})\psi \rangle &= \\ &= \int dq dq' \bar{\chi}(q) \bar{\psi}(q) e^{-\frac{1}{8\alpha} (\frac{4\alpha\beta}{\hbar^2} - 1)(q-q')^2} \chi(q') \psi(q'). \end{aligned} \quad (37)$$

Let us prove that $\langle \psi, \mathcal{Q}_\hbar(f_{\alpha, \beta}^{x_0})\psi \rangle \geq 0 \forall \psi$ if and only if $\alpha\beta \geq (\hbar/2)^2$. The conclusion is obvious for $\alpha\beta = (\hbar/2)^2$. For $\alpha\beta > (\hbar/2)^2$ the conclusion follows from the fact that, in this case, the gaussian function in the integrand can be written as the Fourier transform of a gaussian measure. It remains to show that positivity fails for $\alpha\beta < (\hbar/2)^2$, i.e. that one can in this case find $\psi \in L^2(\mathbb{R})$ such that $\langle \psi, \mathcal{Q}_\hbar(f_{\alpha, \beta}^{x_0})\psi \rangle < 0$. To prove it, let us consider the family of vectors

$$\psi_\sigma(q) = (q - q_0) e^{-\frac{1}{2\sigma}(q-q_0)^2} e^{\frac{ip_0}{\hbar}q}, \quad (38)$$

²See [5] and [6] for a general discussion.

with $\sigma > 0$. From (36) and (37) we obtain

$$\begin{aligned} \langle \psi_\sigma, \mathcal{Q}_\hbar(f_{\alpha,\beta}^{x_0}) \psi_\sigma \rangle &= \\ &= \left(\frac{2\beta}{\pi\hbar^2} \right)^{1/2} \int dq_1 dq_2 q_1 q_2 e^{-\frac{1}{2}(\frac{1}{\sigma} + \frac{1}{2\alpha})(q_1^2 + q_2^2)} \cdot e^{-\frac{1}{2}\Theta(q_1^2 + q_2^2 - 2q_1 q_2)}, \end{aligned} \quad (39)$$

where

$$q_1 := q - q_0, \quad q_2 := q' - q_0, \quad \Theta := \frac{1}{4\alpha} \left(\frac{4\alpha\beta}{\hbar^2} - 1 \right). \quad (40)$$

Let us choose σ such that

$$\frac{1}{\sigma} + \frac{1}{2\alpha} + 2\Theta \neq 0. \quad (41)$$

One can then write (39) as a gaussian integral in \mathbb{R}^2 :

$$\langle \psi_\sigma, \mathcal{Q}_\hbar(f_{\alpha,\beta}^{x_0}) \psi_\sigma \rangle = \left(\frac{2\beta}{\pi\hbar^2} \right)^{1/4} \frac{2\pi}{D} \int \frac{dq_1 dq_2}{(2\pi/D)} q_1 q_2 e^{-\frac{1}{2}(q_1 \ q_2) C^{-1} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}}, \quad (42)$$

where C is the 2×2 matrix such that

$$C^{-1} := \begin{pmatrix} \frac{1}{\sigma} + \frac{1}{2\alpha} + \Theta & -\Theta \\ -\Theta & \frac{1}{\sigma} + \frac{1}{2\alpha} + \Theta \end{pmatrix} \quad (43)$$

and

$$D := \det C^{-1} = \left(\frac{1}{\sigma} + \frac{1}{2\alpha} \right) \left(\frac{1}{\sigma} + \frac{1}{2\alpha} + 2\Theta \right). \quad (44)$$

The gaussian integral (42) is now trivial:

$$\int \frac{dq_1 dq_2}{(2\pi/D)} q_1 q_2 e^{-\frac{1}{2}(q_1 \ q_2) C^{-1} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}} = C_{12} = \frac{\Theta}{D}. \quad (45)$$

Putting it all together we finally get

$$\langle \psi_\sigma, \mathcal{Q}_\hbar(f_{\alpha,\beta}^{x_0}) \psi_\sigma \rangle = \left(\frac{2\beta}{\pi\hbar^2} \right)^{1/4} \frac{2\pi}{D^2} \Theta. \quad (46)$$

Since one can obviously choose $\sigma > 0$ compatible with (41) and $\Theta < 0$, it is clear that the operator $\mathcal{Q}_\hbar(f_{\alpha,\beta}^{x_0})$, associated with the positive observable $f_{\alpha,\beta}^{x_0}$, is not positive for $\frac{4\alpha\beta}{\hbar^2} < 1$.

It is interesting to analyse this lack of positivity in the W-M quantization in light of Heisenberg's uncertainty relations. Note first that the observable $f_{\alpha,\beta}^{x_0}$ (34) with $\alpha\beta = (\hbar/2)^2$ is mapped precisely to the projector (35) onto the quantum state χ (36). This allows the semiclassical interpretation of $f_{\alpha,\beta}^{x_0}$ with $\alpha\beta = (\hbar/2)^2$ as the “characteristic function of the quantum state centered at x_0 ”. Less peaked gaussian functions, i.e. with $\alpha\beta > (\hbar/2)^2$ and therefore with a slow variation with respect to the quantum scale $\hbar/2$ are “well quantized”, i.e. they are mapped to positive operators. When it comes down to gaussian functions that probe regions of the phase space of area less than $\hbar/2$ (which is the lower limit of the uncertainty relations), positivity is lost. Note however that, for $\alpha\beta < (\hbar/2)^2$, there is no physical reason to require correspondence between (in particular the spectrum of) the operator $\mathcal{Q}_\hbar(f_{\alpha,\beta}^{x_0})$ and the observable $f_{\alpha,\beta}^{x_0}$. In fact, precisely because those functions probe deep inside intrinsically quantum domains in phase space, they are inaccessible to the classical observer. The operators $\mathcal{Q}_\hbar(f_{\alpha,\beta}^{x_0})$ in question, if they are true physical observables, which is questionable, certainly have no classical limit.

4 Strict quantization

Given a physical system admitting a classical mechanics description, by *quantization* one means finding a “corresponding” quantum description. By hypothesis, the system in question exhibits, under certain physical conditions determined by the values of the physical observables involved, a classical mechanical behaviour. The correspondence between the classical model and the quantum model is established at this limit: the predictions of the classical model should be a good approximation to the predictions of the “true quantum theory” at the classical regime, i.e. when the system evolves subjected to classical physical conditions.

Although reasonably clear from the conceptual point of view, the establishment of the classical limit of a quantum theory is also a complex problem, given the substantial differences between the formalisms of the two models, classical and quantum.

In this sense, the emphasis on algebraic aspects constitutes a step towards the formal approximation of the two models, useful both in the question of the classical limit and in the inverse problem, that of quantization.

Let us then consider a physical system, with which we associate a Poisson manifold \mathcal{P} and an Hilbert space \mathcal{H} . As discussed in section 2, we assume as chosen a complete algebra of regular classical observables $\mathcal{A}(\mathcal{P})$. Given that the quantum and the classical model describe the same system, there should be a correspondence between functions $f \in \mathcal{A}(\mathcal{P})$ and quantum operators, let's say $\mathcal{Q}(f)$, having f a classical limit. One expects the operators $\mathcal{Q}(f)$, $f \in \mathcal{A}(\mathcal{P})$, to be bounded, and therefore \mathcal{Q} should be a map between $\mathcal{A}(\mathcal{P})$ and $B(\mathcal{H})$, required to be linear and real, i.e., $\mathcal{Q}(\bar{f}) = \mathcal{Q}(f)^+$. The algebra of quantum observables is thus assumed to be $B(\mathcal{H})$, which is obviously complete, in the sense that it acts irreducibly on \mathcal{H} .

In this context, the observer deals with two algebras of observables: the classical algebra, fitting phenomena at the classical scale; and the quantum algebra, describing, in principle, phenomena at any scale. A viable and useful perspective consists in admitting the existence of a continuous family of algebras, interpolating between the classical and the fully quantum domains. This is in broad terms the quantization programme put forward by Rieffel [13] and Landsman [6], leading to the following definition.

Definition 6 *Let \mathcal{P} be a Poisson manifold and $\mathcal{A}(\mathcal{P})$ a complete algebra of regular observables on \mathcal{P} . Let $\mathcal{I} \subset \mathbb{R}$ be a set containing zero as a limit point. A strict quantization of $\mathcal{A}(\mathcal{P})$, labeled by \mathcal{I} , is a family of pairs $\{(A_{\hbar}, \mathcal{Q}_{\hbar})\}_{\hbar \in \mathcal{I}}$, where each A_{\hbar} is a C^* -algebra and each \mathcal{Q}_{\hbar} is a linear map $\mathcal{Q}_{\hbar} : \mathcal{A}(\mathcal{P}) \rightarrow A_{\hbar}$, with $A_0 = C_0(\mathcal{P})$, $\mathcal{Q}_0(f) = f$, $\forall f \in \mathcal{A}(\mathcal{P})$, such that:*

- (i) $\mathcal{Q}_{\hbar}(\bar{f}) = \mathcal{Q}_{\hbar}^+(f)$, $\forall \hbar \in \mathcal{I}$, $\forall f \in \mathcal{A}(\mathcal{P})$.
- (ii) $\lim_{\hbar \rightarrow 0} \|\mathcal{Q}_{\hbar}(\{f, g\}) - [\mathcal{Q}_{\hbar}(f), \mathcal{Q}_{\hbar}(g)]_{\hbar}\| = 0$.
- (iii) $\lim_{\hbar \rightarrow 0} \|\mathcal{Q}_{\hbar}(fg) - \mathcal{Q}_{\hbar}(f) \circ \mathcal{Q}_{\hbar}(g)\| = 0$.
- (iv) $\lim_{\hbar \rightarrow 0} \|\mathcal{Q}_{\hbar}(f)\| = \|f\|$.

Conditions (ii) to (iv) establish the sense in which the classical limit is understood or, from the point of view of quantization, the conditions that the maps \mathcal{Q}_{\hbar} should fulfil in order to ensure correspondence with the classical theory. Condition (iii) replaces the so-called von Neumann condition on the preservation of the multiplicative structure. Condition

(ii) is the implementation, in this formalism, of Dirac's original idea that the quantum correspondent of the Poisson bracket is the quantum Lie bracket $[\cdot, \cdot]_{\hbar}$. Condition (iv) gives some control over the spectral radius of the operators, ensuring in particular that the quantum spectrum is not radically different from the classical spectrum.

The perfect example of a strict quantization is the Weyl-Moyal quantization. In this particular case, the maps $\mathcal{Q}_{\hbar} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{K}(L^2(\mathbb{R}))$ are bijective and it is therefore possible, using the inverse maps \mathcal{Q}_{\hbar}^{-1} , to transpose the multiplicative structures over to $\mathcal{S}(\mathbb{R})$, i.e., to define a family of C^* -products, say \star_{\hbar} , on $\mathcal{S}(\mathbb{R})$, thus obtaining a deformation of the commutative algebra $\mathcal{A}(\mathcal{P})$.

Definition 7 *A strict quantization $\{(A_{\hbar}, \mathcal{Q}_{\hbar})\}_{\hbar \in \mathcal{I}}$ is said to be a strict deformation quantization if $\mathcal{Q}_{\hbar}(A_0)$ is a subalgebra and the maps \mathcal{Q}_{\hbar} are injective.*

5 Smooth groupoids

Some interesting deformations of classical algebras, and in particular Landsman's quantization of the cotangent bundle of a Riemannian manifold, are naturally associated with groupoid convolution algebras. We review here very briefly the necessary notions, following [10] and [8].

A groupoid G can be seen as a generalization of the notion of group. In a group every element can be combined with each other, i.e. there is a map $G \times G \rightarrow G$. In a groupoid one drops the hypothesis that the map is defined for every pair of elements; it is only assumed the existence of a binary operation on a subset, say $G^{(2)}$, of $G \times G$.

Definition 8 *A groupoid is a (concrete) category G such that all the arrows in the category have an inverse. The elements of the groupoid are the arrows of the category, the composition of which defines the binary operation on the groupoid.*

We present next some examples of groupoids. In what follows, we identify the set $\text{Obj}G$ of objects of the category with the set of identity arrows and denote by G both the category and the set $\text{Mor}G$ of its morphisms, or arrows. We use still the following notation: $\text{Hom}[x, y]$ denotes the set of morphisms from x to y ; s (resp. r) denotes the map that applies $g \in \text{Hom}[x, y]$ into x (resp. y).

Example 1. A group is a category with the identity as the only object. The elements of the group are the arrows of the category, composition being the group operation. Since all arrows are invertible, a group is a groupoid.

Example 2. A vector bundle (E, V, M, Π) with fiber V over a manifold M is a category whose objects are the points of M . The arrows of the category are the elements of V at each point of M , i.e., $\text{Hom}[x, y] = \emptyset$ if $x \neq y$ and $\text{Hom}[x, x] = \Pi_x E \cong V$. Composition of arrows is defined by vector sum in the fiber, i.e., $(x, X)(x, Y) = (x, X + Y)$, $x \in M$, $X, Y \in V$.

Example 3. Given a set X , the product $X \times X$ is a groupoid with the following category structure: the objects of the category are the points of X , the arrows are the elements of $X \times X$, i.e. $\text{Hom}[y, x] = \{(x, y)\}$. The composition of arrows is given by $(x, y)(y, z) = (x, z)$.

Example 4. Given a set X , a group Γ and a right action $\alpha : X \times \Gamma \rightarrow X$, one can obtain the semidirect product groupoid $G = X \rtimes \Gamma$ as follows. The set of objects coincides

with X . The arrows constitute the set $X \times \Gamma$, with $(x, \gamma) \in \text{Hom}[\alpha_\gamma(x), x]$. Combination of arrow is given by: $(x, \gamma_1)(y, \gamma_2) = (x, \gamma_1\gamma_2)$, if $\alpha_{\gamma_1}(x) = y$. The inverse of the arrow (x, γ) is $(\alpha_\gamma(x), \gamma^{-1})$.

Let us now consider the introduction of a compatible differential structure on a groupoid [10].

Definition 9 *A smooth groupoid is a groupoid G such that:*

- (i) G , $\text{Obj}G$ and the set $G^{(2)} \subset G \times G$ of pairs of combinable arrows are smooth manifolds.
- (ii) The inclusion $\text{Obj}G \rightarrow G$, the composition of arrows $G^{(2)} \rightarrow G$ and the inversion of arrows are smooth maps.
- (iii) The maps $r, s : G \rightarrow \text{Obj}G$ are submersions.

One can now construct convolution algebras and finally a C^* -algebra associated with a smooth groupoid and a family of measures, as follows.

Let then G be a smooth groupoid. For each $x \in \text{Obj}G$, consider the sets $G^x := \cup_{y \in \text{Obj}G} \text{Hom}[y, x]$ and $G_x := \cup_{y \in \text{Obj}G} \text{Hom}[x, y]$, called r -fibre and s -fibre, respectively. These fibres inherited a locally compact topology, induced from the manifold structure of G [10].

For each $g \in G$ there is a map $\Theta_g : G^{s(g)} \rightarrow G^{r(g)}$, defined by $\Theta_g(g') = gg'$, which establishes a bijection between $G^{s(g)}$ and $G^{r(g)}$ (since every arrow g is invertible).

A family of measures μ^x on the r -fibres G^x is said to be a *Haar system* if it satisfies the compatibility conditions $\mu^{r(g)} = (\Theta_g)_* \mu^{s(g)}$, $\forall g \in G$, where $(\Theta_g)_* \mu^{s(g)}$ is the push-forward of the measure $\mu^{s(g)}$, with respect to the map Θ_g . Finally, note that the map $g \mapsto g^{-1}$ establishes also a bijection between G^x and G_x , for every $x \in \text{Obj}G$. By means of these bijections, a Haar system defines also a family of measures on the s -fibres G_x .

Definition 10 *Let G be a smooth groupoid equipped with a Haar system of measures. The following defines a convolution on the space of C^∞ functions on G with compact support:*

$$(F_1 \star F_2)(g) = \int_{G^{r(g)}} F_1(h) F_2(h^{-1}g) d\mu^{r(g)}(h), \quad F_1, F_2 \in C_c^\infty(G). \quad (47)$$

On the same space, an involution is defined by

$$F(g) \mapsto \bar{F}(g^{-1}), \quad F \in C_c^\infty(G). \quad (48)$$

The convolution algebra $C_c^\infty(G)$ admits natural $*$ -representations, one per each $x \in \text{Obj}G$. In fact, let us consider the Hilbert spaces $L^2(G_x, \mu_x)$, $x \in \text{Obj}G$. The (involutive) representations $(L^2(G_x, \mu_x), \pi_x)$ are defined by

$$(\pi_x(F)\psi)(g) = \int_{G^{r(g)}} F(h)\psi(h^{-1}g) d\mu^{r(g)}(h), \quad (49)$$

where $F \in C_c^\infty(G)$, $\psi \in L^2(G_x, \mu_x)$ and $g \in G_x$.

Definition 11 *Let G be a smooth groupoid with Haar system $\{\mu^x\}_{x \in \text{Obj}G}$. The associated (reduced) C^* -algebra $C_r^*(G)$ is the completion of the convolution algebra $C_c^\infty(G)$ with respect to the norm $\|F\| := \sup_{x \in \text{Obj}G} \|\pi_x(F)\|$.*

Let us analyse again the previous examples, each of which groupoid is now equipped with a natural differential structure.

Example 1a. In a group there is only one object, and therefore we have only one r -fibre and one s -fibre, both coincident with the group itself. In this case a locally compact topology is sufficient to construct the C^* -algebra, which coincides with the convolution algebra on the group for a given Haar measure. Consider in particular the additive group \mathbb{R} with the Lebesgue measure: the obtained C^* -algebra is the Fourier transform of the multiplicative algebra $C_0(\mathbb{R})$.

Example 2a. We consider the particular case of a tangent bundle TM of a n -dimensional Riemannian manifold (M, \mathbf{g}) . Let us fix a local coordinate system (q^1, \dots, q^n) in M , collectively denoted by q . At each point $q \in M$ the r -fibre G^q coincides with $T_q M$, which in turn can be identified with \mathbb{R}^n by $X = (X^1, \dots, X^n) \mapsto \sum X^i \partial_{q^i}$. On the r -fibres we consider the measure $d\mu^q(X) = \sqrt{\det \mathbf{g}(q)} d^n X$, where $d^n X$ is the Lebesgue measure and \mathbf{g} is the metric. Convolution is given by integration on the fibre at each point of M :

$$(F_1 \star F_2)(q, X) = \int_{T_q M} F_1(q, Y) F_2(q, X - Y) d\mu^q(Y). \quad (50)$$

The obtained C^* -algebra is isomorphic to the multiplicative algebra $C_0(T^*M)$, by Fourier transform \mathcal{F} on the fibre:

$$(\mathcal{F}F)(q, \xi) := \int_{T_q M} e^{-i\xi X} F(q, X) d\mu^q(X), \quad (51)$$

where $(q, \xi) \in T^*M$, $\mathcal{F}F \in C_0(T^*M)$.

Example 3a. Let (M, \mathbf{g}) be a n -dimensional Riemannian manifold and consider the groupoid $M \times M$. Every r -fibre and s -fibre is isomorphic to M . On each fibre, we consider the measure $d\mu(q) = \sqrt{\det \mathbf{g}(q)} d^n q$, for some local coordinate system on M . The convolution is

$$(F_1 \star F_2)(q, q') = \int_{G^q} F_1(q, q'') F_2((q, q'')^{-1}(q, q')) d\mu(q'') \quad (52)$$

$$= \int_M F_1(q, q'') F_2(q'', q') d\mu(q''), \quad (53)$$

where one can recognize immediately the convolution of kernels of integral operators in $L^2(M, \mu)$. In fact, the algebra $C_r^*(M \times M)$ is isomorphic to the algebra $\mathcal{K}(L^2(M, \mu))$ of compact operators in $L^2(M, \mu)$ [7].

Example 4a. In this case we analyse an example directly related to the Weyl-Moyal quantization. Consider a family of actions α^ϵ of \mathbb{R} on \mathbb{R} , labeled by a real number ϵ . For each ϵ the actions are $\alpha_y^\epsilon(x) = x + \epsilon y$. Independently of ϵ , the r -fibres and s -fibres of the semidirect product $\mathbb{R} \rtimes_{\alpha^\epsilon} \mathbb{R}$ can be identified with \mathbb{R} . Let us then introduce the Lebesgue measure on each fiber. The convolution is then given by

$$(f \star g)(x, y) = \int_{G^x} f(x, z) g((x, z)^{-1}(x, y)) dz \quad (54)$$

$$= \int_{\mathbb{R}} f(x, z) g((x + \epsilon z, -z)(x, y)) dz \quad (55)$$

$$= \int_{\mathbb{R}} f(x, z) g(x + \epsilon z, y - z) dz. \quad (56)$$

Concerning the action of the elements of the algebra on the Hilbert spaces $L^2(G_u)$, let us distinguish the cases $\epsilon = 0$ and $\epsilon \neq 0$. For $\epsilon > 0$ the action does not depend on the s -fibre. It is defined on $L^2(\mathbb{R})$ by

$$(\pi(f)\psi)(x) = \int_{\mathbb{R}} f(x, y)\psi(x + \epsilon y)dy, \quad \psi \in L^2(\mathbb{R}). \quad (57)$$

For $\epsilon = 0$ we get the representations $\{(\pi_u, L^2(\mathbb{R}))\}_{u \in \mathbb{R}}$:

$$((\pi_u f)\psi)(x) = \int_{\mathbb{R}} f(u, y)\psi(x - y)dy. \quad (58)$$

We recognize for $\epsilon = 0$ the Fourier transform (in the second variable) of the multiplicative algebra $C_0(\mathbb{R} \times \mathbb{R})$. In the case $\epsilon > 0$ the nontrivial action of \mathbb{R} on \mathbb{R} deforms the convolution algebra in a way that corresponds precisely to the Weyl-Moyal deformation, see section 7.2 below.

6 Tangent groupoid

This section is dedicated to Landsman's quantization of the cotangent bundle of a Riemannian manifold Q [16]. Although this construction can be described without reference to Connes' tangent groupoid [10], we adhere to the tangent groupoid perspective right from the start, since it is quite natural, both from the geometric and the algebraic viewpoints. We start by showing how the simplest case, that of $Q = \mathbb{R}$, fits in this framework.

6.1 Weyl-Moyal quantization revisited

The Weyl-Moyal quantization admits a reformulation in terms of the so-called tangent groupoid. As we will see shortly, the classical algebra and the quantum algebras appear unified, as elements of the same algebra of functions on the tangent groupoid.

First note that the W-M quantization maps \mathcal{Q}_\hbar (21) can be naturally split into three distinct maps. Consider the Fourier transform

$$\mathcal{F} : \mathcal{S}(T^*\mathbb{R}) \rightarrow \mathcal{S}(T\mathbb{R}) \quad (59)$$

$$f(q, p) \mapsto \tilde{f}(q, v) = \int \frac{dp}{2\pi} e^{ipv} f(q, p) \quad (60)$$

and the representation $\pi : \mathcal{S}(\mathbb{R} \times \mathbb{R}) \rightarrow \mathcal{K}(L^2(\mathbb{R}))$ of $\mathcal{S}(\mathbb{R} \times \mathbb{R})$ functions as kernels of integral operators. It is then clear that the maps $\mathcal{Q}_\hbar : \mathcal{S}(T^*\mathbb{R}) \rightarrow \mathcal{K}(L^2(\mathbb{R}))$ (21) are obtained as the composition

$$\mathcal{Q}_\hbar = \pi \circ \varphi_\hbar \circ \mathcal{F}, \quad (61)$$

where the map $\varphi_\hbar : \mathcal{S}(T\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R} \times \mathbb{R})$ is defined by

$$(\varphi_\hbar \tilde{f})(x, y) = \frac{1}{\hbar} \tilde{f}\left(\frac{x+y}{2}, \frac{x-y}{\hbar}\right). \quad (62)$$

The importance of this decomposition is that it isolates the nontrivial map φ_\hbar , in which is effectively present the deformation of the algebraic structure. In fact, the Fourier transform is a morphism from the multiplicative algebra $\mathcal{S}(T^*\mathbb{R})$ to the convolution algebra (with respect to the second variable) in $\mathcal{S}(T\mathbb{R})$. This is, in turn, the algebra naturally associated

with the groupoid structure of $T\mathbb{R}$. As we have seen above, $\mathbb{R} \times \mathbb{R}$ is also a groupoid, and the map π is precisely a morphism from the associated groupoid algebra to the algebra $\mathcal{K}(L^2(\mathbb{R}))$ of compact operators.

The W-M maps are therefore naturally decomposed into a couple of morphisms and a map between groupoid algebras, the deformation φ_{\hbar} :

$$\begin{array}{ccc} \mathcal{S}(T^*\mathbb{R}) & \xrightarrow{\mathcal{Q}_{\hbar}} & \mathcal{K}(L^2(\mathbb{R})) \\ \mathcal{F} \downarrow & & \uparrow \pi \\ C_r^*(T\mathbb{R}) & \xrightarrow{\varphi_{\hbar}} & C_r^*((\mathbb{R} \times \mathbb{R}) \times \{\hbar\}) , \end{array}$$

where $C_r^*(T\mathbb{R})$ denotes the C^* -algebra of the groupoid $T\mathbb{R}$ and $C_r^*((\mathbb{R} \times \mathbb{R}) \times \{\hbar\})$ denotes the C^* -algebra of the groupoid $(\mathbb{R} \times \mathbb{R}) \times \{\hbar\} \cong \mathbb{R} \times \mathbb{R}$.

The crucial point is that the transformations φ_{\hbar} are induced by an identification of $T\mathbb{R}$ with $\mathbb{R} \times \mathbb{R}$. Let us consider the map $\phi : T\mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ defined by

$$T\mathbb{R} \ni (q, v) \mapsto (q + \frac{1}{2}v, q - \frac{1}{2}v) . \quad (63)$$

Consider still the family of maps $\phi_{\hbar} : T\mathbb{R} \rightarrow (\mathbb{R} \times \mathbb{R}) \times \{\hbar\}$ obtained from the previous one:

$$\phi_{\hbar}(q, v) = \phi(q, \hbar v) = (q + \frac{\hbar}{2}v, q - \frac{\hbar}{2}v) . \quad (64)$$

It is then clear that

$$(\varphi_{\hbar} \tilde{f})(x, y) = \frac{1}{\hbar} \tilde{f}(\phi_{\hbar}^{-1}(x, y)) . \quad (65)$$

The maps ϕ_{\hbar} (64) allow the construction of a manifold with boundary, the so-called tangent groupoid [10], as follows. Consider first the product manifold $G_{\mathbb{R}}^1 := (\mathbb{R} \times \mathbb{R}) \times]0, 1]$. This is also a groupoid: two elements (x, y, \hbar) and (x', y', \hbar') can be combined if and only if they belong to the same leaf $(\mathbb{R} \times \mathbb{R}) \times \{\hbar\}$, i.e. if $\hbar = \hbar'$. The associated C^* -algebra turns out to be $C_r^*(G_{\mathbb{R}}^1) \cong C_0([0, 1]) \otimes C_r^*(\mathbb{R} \times \mathbb{R}) \cong C_0([0, 1]) \otimes \mathcal{K}(L^2(\mathbb{R}))$. It is interesting to note that $G_{\mathbb{R}}^1$ can be seen as the space of secant lines to \mathbb{R} , or more precisely of finite difference operators. In fact, the elements $(x, y, \hbar) \in G_{\mathbb{R}}^1$ define elements of the dual of the space $C^1(\mathbb{R})$ of differentiable functions in \mathbb{R} , by

$$(x, y, \hbar) \mapsto \frac{f(x) - f(y)}{\hbar} , \quad f \in C^1(\mathbb{R}) .$$

The closure of this open set in $C^1(\mathbb{R})^*$ is the tangent groupoid [10, 17].

An explicit construction of the tangent groupoid, both as a manifold and as a groupoid, is the following [10] (see also [17, 7]). Let us consider the union $G_{\mathbb{R}} := G_{\mathbb{R}}^1 \cup G_{\mathbb{R}}^2$, where $G_{\mathbb{R}}^2 := T\mathbb{R}$. $G_{\mathbb{R}}$ is a groupoid with the obvious structure of union of groupoids. The structure of manifold with boundary is defined by the maps ϕ_{\hbar} (64), which give coordinates in $G_{\mathbb{R}}$. In fact, as a manifold, the tangent groupoid $G_{\mathbb{R}}$ is diffeomorphic to $T\mathbb{R} \times [0, 1]$. The announced coordinate system in $G_{\mathbb{R}}$ is given by the transformation

$$\Phi : T\mathbb{R} \times [0, 1] \rightarrow G_{\mathbb{R}} \quad (66)$$

such that

$$\Phi(q, v, \hbar) = \begin{cases} (q + \frac{\hbar}{2}v, q - \frac{\hbar}{2}v, \hbar) & \text{if } \hbar > 0 \\ (q, v) & \text{if } \hbar = 0. \end{cases}$$

The transformation Φ is a diffeomorphism when restricted to $T\mathbb{R} \times]0, 1]$ and maps the boundary of $T\mathbb{R} \times [0, 1]$ to the boundary of $G_{\mathbb{R}}$.

The C^* -algebra $C_r^*(G_{\mathbb{R}})$ associated with the tangent groupoid $G_{\mathbb{R}}$ is formed by pairs $(\{k_h\}_{h \in [0, 1]}, \tilde{f})$, where $k_h \in C_0(\mathbb{R} \times \mathbb{R})$, $\tilde{f} \in C_0(T\mathbb{R})$, subject to the continuity condition at the boundary, i.e.

$$\lim_{h \rightarrow 0} k_h(q + \frac{\hbar}{2}v, q - \frac{\hbar}{2}v) = \tilde{f}(q, v) . \quad (67)$$

This continuity condition is in fact a quantization condition, imposing that any element of $C_r^*(G_{\mathbb{R}})$ is a family of quantum operators having $\tilde{f}(q, v)$ (or its inverse Fourier transform $f(q, p)$) as a limit.

The above continuity condition at the boundary still leaves a great deal of freedom as to the choice of quantum operators to be associated with a given classical observable, and therefore the quantization maps are not fixed. However, the construction of the groupoid itself suggests the construction of well determined linear maps $\mathcal{S}(T\mathbb{R}) \rightarrow C_0(\mathbb{R} \times \mathbb{R})$. Let then $\tilde{f}(q, v)$ be an element of $\mathcal{S}(T\mathbb{R})$. Consider the element $\tilde{F} := \{\tilde{f}_h\}_{h \in [0, 1]}$ of $C_r^*(T\mathbb{R} \times [0, 1])$ given by $\tilde{f}_h = \tilde{f}$, $\forall h$. The element $\frac{1}{\hbar}(\Phi^{-1})^*\tilde{F}$ of $C_r^*(G_{\mathbb{R}})$ provides then the required quantization. The composition of this map with the Fourier transform and the representation π finally gives the Weyl-Moyal quantization.

6.2 Cotangent bundle of a Riemannian manifold

Following Landsman [6, 16], Connes [10] and also reference [17], we present in this section the strict quantization of the most common type of phase space in physical applications, which is the cotangent bundle T^*Q of some Riemannian manifold Q . We start with the construction of the tangent groupoid G_Q , which generalizes the construction of the previous section. We introduce first the algebraic structure of the tangent groupoid G_Q , followed by its manifold structure.

Let then (Q, g) be a n -dimensional Riemannian manifold, with metric g . The associated tangent groupoid is a disjoint union of groupoids, $G_Q = ((Q \times Q) \times]0, 1]) \dot{\cup} TQ$, formed by the groupoid TQ and by a family of copies of the groupoid $Q \times Q$. The tangent bundle TQ is a smooth groupoid, equipped with a Haar system of measures $d\mu^q$ on the fibres T_qQ :

$$d\mu^q(v) = d^n v \sqrt{\det g(q)} , \quad (68)$$

where (q, v) denotes local coordinates on TQ . The convolution algebra of TQ is determined by the expression

$$(f \star g)(q, v) = \int d\mu^q(v') f(q, v') g(q, v - v') . \quad (69)$$

The product $Q \times Q$ is also a smooth groupoid, with measure

$$d\mu(q) = d^n q \sqrt{\det g(q)} \quad (70)$$

on the fiber Q . The product $(Q \times Q) \times]0, 1]$ is again a smooth groupoid, with the product manifold structure and the following groupoid structure. The elements (x, y, \hbar) and (x', y', \hbar') can be combined if and only if $\hbar = \hbar'$ and $y = x'$, and in that case $(x, y, \hbar)(y, y', \hbar) = (x, y', \hbar)$. The r -fibres and s -fibres of $(Q \times Q) \times]0, 1]$ both coincide with $Q \times \{\hbar\} \cong Q$, and the convolution algebra is given by

$$(f \star g)(x, y, \hbar) = \int d\mu(z) f(x, z, \hbar) g(x, y, \hbar) . \quad (71)$$

Finally, the disjoint union $G_Q = ((Q \times Q) \times]0, 1]) \cup TQ$ acquires a natural groupoid structure, in the sense that elements of $(Q \times Q) \times]0, 1]$ (resp. TQ) combine only amongst themselves.

As in the previous section, which corresponds to $Q = \mathbb{R}$, G_Q becomes a smooth groupoid [10, 17, 7] when equipped with the topology of a manifold with boundary. Generalizing the previous procedure, we present next a map from an open set $U \subset TQ \times]0, 1]$ to G_Q , which defines the boundary of G_Q .

Let us fix a local coordinate system q on Q . This gives us also coordinates (q, q') on $Q \times Q$ and a basis $(\partial_q, \partial_{q'})$ on each space $T_{(q, q')}(Q \times Q)$. Consider the diagonal embedding $\Delta : Q \rightarrow Q \times Q$ given by $\Delta(q) = (q, q)$. At each point $\Delta(q)$ the metric $\mathbf{g} \oplus \mathbf{g}$ on $Q \times Q$ allows a decomposition of $T_{(q, q)}(Q \times Q)$ vectors in tangent and normal parts, i.e.

$$T_{(q, q)}(Q \times Q) = \Delta_* T_q Q \oplus (\Delta_* T_q Q)^\perp,$$

where $\Delta_* T_q Q$ is the push-forward of the tangent space and $(\Delta_* T_q Q)^\perp$ is its orthogonal complement. The union $\cup_{q \in Q} (\Delta_* T_q Q)^\perp$ is a subbundle of the restriction of $T(Q \times Q)$ to $\Delta(Q)$, whose fibres are normal to $\Delta(q)$ at each point. This is the normal bundle associated with Δ , hereafter denoted by $N^\Delta Q$.

Clearly, the elements of $\Delta_* T_q Q$ are of the form (X_q, X_q) , with $X_q \in T_q Q$ and in the same way the elements of $(\Delta_* T_q Q)^\perp$ can be written in the form $(X_q, -X_q)$, with $X_q \in T_q Q$. One can therefore build a map $\eta : TQ \rightarrow N^\Delta Q$, given by

$$\eta(q, X_q) = \left(\Delta(q), \frac{1}{2}X_q, -\frac{1}{2}X_q \right). \quad (72)$$

The transformation η can now be combined with the exponential map defined by normal geodesics at $\Delta(q)$. Let then W_1 be an open set in $N^\Delta Q$ where the exponential map is defined and let $U_1 \subset Q \times Q$ denote the image of W_1 . Consider the transformations $\phi : V_1 \rightarrow U_1$, $\phi = \exp \circ \eta$, where $V_1 := \eta^{-1}(W_1) \in TQ$. Explicitly

$$TQ \ni (q, X_q) \xrightarrow{\eta} \left(\Delta(q), \frac{1}{2}X_q, -\frac{1}{2}X_q \right) \xrightarrow{\exp} \left(\exp_q\left(\frac{1}{2}X_q\right), \exp_q\left(-\frac{1}{2}X_q\right) \right). \quad (73)$$

Let us define still the maps

$$\phi_\hbar : V_\hbar := \frac{1}{\hbar}V_1 \rightarrow V_1 \xrightarrow{\phi} U_\hbar \cong U_1 \times \{\hbar\} \quad (74)$$

$$TQ \ni (q, X_q) \mapsto (q, \hbar X_q) \mapsto \left(\exp_q\left(\frac{1}{2}\hbar X_q\right), \exp_q\left(-\frac{1}{2}\hbar X_q\right), \hbar \right). \quad (75)$$

Finally, consider the manifold $TQ \times [0, 1]$ with its product structure and its boundary $TQ \times \{0\} \cong TQ$. Let U be the open set in $TQ \times [0, 1]$ defined by $U := (X_{\hbar \in]0, 1]} V_\hbar) \times (TQ \times \{0\})$, which contains the boundary. The transformation $\Phi : U \rightarrow G_Q$ defined by

$$\Phi(q, X_q, \hbar) = \begin{cases} \phi_\hbar(q, X_q) & \text{if } \hbar > 0 \\ (q, X_q) & \text{if } \hbar = 0 \end{cases}$$

is a diffeomorphism when restricted to $TQ \times]0, 1]$ and maps TQ to TQ , thus defining a coordinate system on an open set in G_Q containing the boundary. This concludes the description of the tangent groupoid G_Q .

Let us then describe the quantization of the symplectic manifold $\mathcal{P} := T^*Q$. Consider first the measure

$$d\mu_q(p) = \frac{d^n p}{(2\pi)^n \sqrt{\det \mathbf{g}(q)}} \quad (76)$$

on the fibres T_q^*Q of T^*Q , where (q, p) is a local coordinate system. The Fourier transform on the fibre maps functions $f(q, p)$ on T^*Q to functions $\tilde{f}(q, v)$ on TQ :

$$\tilde{f}(q, v) := \int d\mu_q(p) e^{ipv} f(q, p) . \quad (77)$$

Let us adopt as a complete algebra of regular observables $\mathcal{A}(\mathcal{P})$ the subalgebra of the functions $f \in C_0^\infty(\mathcal{P})$ such that $\tilde{f} \in C_c^\infty(TQ)$.

The quantization maps are defined as follows. For a given observable $f \in \mathcal{A}(\mathcal{P})$, let $\hbar(f)$ be a real number such that $\text{supp} \tilde{f} \subset V_{\hbar(f)}$. Then, for every $\hbar \leq \hbar(f)$,

$$K_\hbar^f := \hbar^{-n} (\phi_\hbar^{-1})^* \tilde{f} \quad (78)$$

is well defined and belongs to $C_r^*((Q \times Q) \times \{\hbar\})$. The observable f is therefore quantized, $\forall \hbar \leq \hbar(f)$, by the operator $\mathcal{Q}_\hbar(f) \in \mathcal{K}(L^2(Q, \mu))$:

$$(\mathcal{Q}_\hbar(f)\psi)(x) = \int d\mu(x') K_\hbar^f(x, x') \psi(x'), \quad \psi \in L^2(Q, \mu) . \quad (79)$$

7 The torus: a case study

The 2-torus T^2 provides a good test for any quantization scheme. Although rather innocent looking, the torus possesses a set of characteristics that make it somewhat special. To begin with, it is not a cotangent bundle, and it is compact, and therefore the physical expectation is that at the quantum level one will find only bounded observables and moreover finite dimensional Hilbert spaces. There is, however, another characteristic that distinguishes the torus from e.g. the two-sphere, with which it shares the above two properties. In fact, the Poisson algebra $C^\infty(T^2)$ of the torus does not seem to admit any subalgebra that separates points (and contain the constant function 1), besides the algebra $C^\infty(T^2)$ itself (and dense subalgebras thereof). In particular, it is known that no such finite dimensional subalgebra exists [18]. Thus, from the point of view of canonical quantization, it seems that for T^2 one is forced to impose the Dirac condition (5) on the whole Poisson algebra. But it is also known that no nontrivial finite dimensional Lie representation of $C^\infty(\mathcal{P})$ can be found, for any connected compact symplectic manifold \mathcal{P} [19]. On the other hand, any infinite dimensional representation of such a Poisson algebra will include unbounded operators [20]. So, it seems that every conceivable Dirac-like quantization of the torus will produce infinite dimensional Hilbert spaces and unbounded observables, conflicting with natural physical expectations.

We discuss next a quantization of $C^\infty(T^2)$ proposed by Gotay, which has the great interest of proving that (irreducible) quantizations of full Poisson algebras can indeed be found. It does not, however, avoids the above mentioned drawbacks. A modified quantization, departing from the exact implementation of Dirac's condition, is already presented in the next section. This presentation follows geometric quantization methods, although at some point a deformation is introduced. The same quantization is discussed in section 7.2, this time showing that it is directly obtained from a group action such as those discussed in example 4 of section 5.

7.1 Geometric quantization of the torus

Let (\mathcal{P}, ω) be a symplectic manifold and $\{, \}$ the corresponding Poisson bracket. Let \mathcal{S} be a Lie-subalgebra of $(C^\infty(\mathcal{P}), \{, \})$, containing the constant function 1. By *prequantization*

of \mathcal{S} it is meant a linear map \mathcal{Q} from \mathcal{S} to self-adjoint operators on a Hilbert space, such that Dirac's condition

$$\mathcal{Q}(\{f, g\}) = [\mathcal{Q}(f), \mathcal{Q}(g)]_{\hbar} \quad \forall f, g \in \mathcal{S} \quad (80)$$

is satisfied and

$$\mathcal{Q}(1) = \mathbf{1}, \quad (81)$$

where $\mathbf{1}$ is the identity operator.

In general, Dirac's condition can be achieved in the full algebra $C^\infty(\mathcal{P})$: given that Hamiltonian vector fields provide an (anti-)representation of the Poisson algebra, one could just adopt the map $\mathcal{Q}(f) = -i\hbar\xi_f$, where ξ_f is the Hamiltonian vector field defined by f , acting on (an appropriate dense domain of) $L^2(\mathcal{P}, \omega^n/n!)$. However, this type of representation always leads to $\mathcal{Q}(1) = 0$, which is not acceptable. The formalism of geometric quantization (see e.g. [21, 22, 23]) corrects this aspect. In this formalism, the quantization map is of the form

$$\mathcal{Q}(f) = f - i\hbar\xi_f + \theta(\xi_f),$$

where $d\theta = \omega$. In general, the 1-form θ is defined only locally, with $\xi_f + \frac{i}{\hbar}\theta(\xi_f)$ being properly interpreted as a covariant derivative on a certain line bundle, which requires $h^{-1}\omega$ to be of integral cohomology class. (In this subsection h is Planck's constant.)

Besides (80) and (81), a true *canonical quantization* is required to satisfy a further set of mathematical physics conditions, one of the most prominent being irreducibility (see [12] for a thorough discussion). It is this irreducibility condition that typically calls for the necessity of a polarization in geometric quantization, leading to a drastic reduction of the algebra over which the quantization map is defined.

Nevertheless, in [24] (see also [12]) Gotay shows that a given prequantization of the full algebra of smooth functions on the torus T^2 satisfies even the condition of being irreducible. Thus, it seems that in this case polarization would not be necessary and that a canonical quantization of *all* smooth observables has been achieved. Despite the obvious interest of this result, there is a high price to pay for this full quantization, in the sense that von Neumann's condition (4) is badly broken (see [25]). We discuss next Gotay's proposal.

Let $N \in \mathbb{N}$ and consider the torus T^2 of area Nh , which we identify with $\mathbb{R}^2/\mathbb{Z}^2$ equipped with the symplectic form

$$\omega = Nh \, dx \wedge dy, \quad (82)$$

where we have introduced local coordinates (x, y) . The Poisson algebra $C^\infty(T^2)$ is therefore the algebra of C^∞ periodic functions on \mathbb{R}^2 . Gotay's quantization, which is obtained through geometric quantization methods, can be described as follows. Choosing the connection $\theta = -Nhydx$, the prequantum Hilbert space \mathcal{H}_N can be seen as the completion of the space \mathcal{D}_N of complex C^∞ functions in \mathbb{R}^2 such that,

$$\phi(x + m, y + n) = e^{2\pi i N n x} \phi(x, y), \quad \forall m, n \in \mathbb{Z}, \quad (83)$$

with respect to the inner product

$$\langle \phi, \phi' \rangle = \int_{[0,1] \times [0,1]} dx dy \, \bar{\phi} \phi'. \quad (84)$$

The prequantization map is given by

$$\mathcal{Q}_N(f)\phi = f\phi - \frac{i}{2\pi N} \left(\frac{\partial f}{\partial y} \left(\frac{\partial \phi}{\partial x} - 2\pi N i y \phi \right) - \frac{\partial f}{\partial x} \frac{\partial \phi}{\partial y} \right), \quad \forall f \in C^\infty(T^2). \quad (85)$$

Being a prequantization, it is true $\forall N$ that Dirac's condition (5) is fulfilled for all observables in the algebra $C^\infty(T^2)$. The case $N = 1$ is special in that irreducibility conditions are satisfied [24]. Thus, it appears that \mathcal{Q}_1 above gives a *bona fide* canonical quantization of *all* observables in a symplectic manifold. However, there is no control over the multiplicative structure of the algebra $C^\infty(T^2)$, and therefore there is also no control over the spectrum of the quantum operators $\mathcal{Q}_1(f)$ (85). For instance, considering the classical observables $\sin(2\pi x)$ and $\cos(2\pi x)$, one can show [25] that

$$\mathcal{Q}_1^2(\sin(2\pi x)) + \mathcal{Q}_1^2(\cos(2\pi x)) = \mathbf{1} + \mathcal{R}, \quad (86)$$

where \mathcal{R} is an *unbounded* operator with no correspondence with any observable. (The same happens with the functions $\sin(2\pi y)$ and $\cos(2\pi y)$.) In particular, the spectrum of the quantum operators corresponding to the *sinus* and *cosinus* functions is the full line \mathbb{R} , and the correlation between the two functions is lost.

Let us now discuss a different quantization of the torus T^2 which we believe satisfies appropriate physical requirements. This quantization can be introduced in a number of ways (see [26, 27, 28, 7]). Our treatment in this section is inspired in [26]. As we will see in the next section, the same quantization appears naturally in the context of noncommutative geometry.

Let us consider then the prequantizations \mathcal{Q}_N (85). In the geometric quantization formalism, the way to achieve a quantization starting from a given prequantization is to restrict the action of observables to (covariantly) constant sections over a given polarization [21, 22]. Let us then focus on the space of sections $\phi \in \mathcal{H}_N$ such that $\frac{\partial}{\partial y}\phi = 0$ (note that with the connection $\theta = -Nhydx$, the covariant derivative $\nabla_y := \frac{\partial}{\partial y} + \frac{i}{\hbar}\theta(\frac{\partial}{\partial y})$ coincides with $\frac{\partial}{\partial y}$). Taking (83) into account, those sections satisfy

$$(1 - e^{2\pi i N x})\psi(x, 0) = 0, \quad (87)$$

and it is therefore clear that there are no nontrivial solutions in \mathcal{H}_N . There are, however N independent generalized solutions, of the form

$$\psi_k = \delta(x - k/N), \quad k = 0, 1, \dots, N-1. \quad (88)$$

In fact, the distributions ψ_k defined by:

$$\psi_k(\phi) = \int dx dy \delta(x - k/N) \phi(x, y) = \int_0^1 dy \phi(k/N, y) \quad (89)$$

are well defined on the dense space $\mathcal{D}_N \subset \mathcal{H}_N$ and satisfy

$$\psi_k\left(\frac{\partial \phi}{\partial y}\right) = 0, \quad \forall \phi \in \mathcal{D}_N. \quad (90)$$

(The appearance of distributional solutions is common in geometric quantization; the fibres $x = k/N$ are an example of so-called Bohr-Sommerfeld submanifolds [21, 22].) Let us then choose the finite dimensional Hilbert space (isomorphic to \mathbb{C}^N) generated by the

N distributions ψ_k , with inner product $\langle \psi_k, \psi_{k'} \rangle = \delta_{kk'}$, to be the quantum Hilbert space associated with the 2-torus of area Nh . In general, the choice of a polarization selects a restricted subalgebra of observables with a well defined action on the quantum Hilbert space. In the present case, one can easily check that a quantum observable $\mathcal{Q}_N(f)$ is well defined if and only if

$$\psi_k \left(\mathcal{Q}_N(f) \frac{\partial \phi}{\partial y} \right) = \psi_k \left(\nabla_x \left(\frac{\partial^2 f}{\partial y^2} \phi \right) \right) = 0, \quad \forall k, \quad \forall \phi. \quad (91)$$

It follows that the only functions $f \in C^\infty(T^2)$ that are quantized by this process are the ones which depend exclusively on x . These act simply by multiplication, i.e.

$$\mathcal{Q}_N(f(x))\psi_k = f(k/N)\psi_k. \quad (92)$$

The extension of the quantization to further observables requires a new look at the quantization of functions $g(y) \in C^\infty(T)$.

Following [26], let us consider the unitary operators given by $\mathcal{V}_N(b)$, $b \in \mathbb{R}$:

$$(\mathcal{V}_N(b)\phi)(x, y) := \phi(x + b, y), \quad \phi \in \mathcal{H}_N, \quad (93)$$

which are associated with translations $x \mapsto x + b$. Clearly, the operators $\mathcal{V}_N(b)$ have a well defined action in \mathcal{H}_N only for values of b of the form $b = n/N$, $n \in \mathbb{Z}$, as follows from (83). These finite translations correspond to maps between the Bohr-Sommerfeld leaves, and we therefore get well defined unitary translation operators $\mathcal{V}_N(n)$ on the quantum Hilbert space:

$$\mathcal{V}_N(n)\psi_k := \psi_{k-n \pmod{N}}. \quad (94)$$

Taking into account the Weyl-Moyal quantization³ in $T^*\mathbb{R}$, one can look at the operators $\mathcal{V}_N(n)$ as the quantization of the functions $e^{2\pi i n y}$. This interpretation is further supported by the commutation relations satisfied by these operators and the quantization of the functions $e^{2\pi i m x}$, namely

$$\mathcal{V}_N(n) \mathcal{Q}_N(e^{2\pi i m x}) = e^{2\pi i m n / N} \mathcal{Q}_N(e^{2\pi i m x}) \mathcal{V}_N(n). \quad (95)$$

Let us then define a quantization map by

$$\mathcal{Q}_N(e^{2\pi i(m x + n y)}) := e^{\pi i m n / N} \mathcal{Q}_N(e^{2\pi i m x}) \mathcal{V}_N(n). \quad (96)$$

Concerning the Dirac rule we obtain in particular

$$\begin{aligned} \left[\mathcal{Q}_N(e^{2\pi i m x}), \mathcal{Q}_N(e^{2\pi i n y}) \right] - i\hbar \mathcal{Q}_N(\{e^{2\pi i m x}, e^{2\pi i n y}\}) &= \\ &= 2i \left(m n \pi / N - \sin(m n \pi / N) \right) \mathcal{Q}_N(e^{2\pi i(m x + n y)}), \end{aligned} \quad (97)$$

which shows that, for large N , the Dirac condition is well approximated by slow varying functions, i.e., such that the Fourier decomposition contains only components $e^{2\pi i(m x + n y)}$ of frequencies m and n which are small compared to N .

³In $T^*\mathbb{R}$, the W-M quantization (21) of the function $g(p) = e^{i b p / \hbar}$ is in fact the translation operator in $L^2(\mathbb{R})$: $\psi(q) \mapsto \psi(q + b)$.

7.2 C^* -algebraic quantization of the torus

The Weyl-Moyal quantization map (21), section 3, can be immediately rewritten in the form

$$(\mathcal{Q}_\hbar(f)\psi)(x) = \int \left(\int \frac{dp}{2\pi} e^{-ipy} f\left(x + \frac{\hbar}{2}y, p\right) \right) \psi(x + \hbar y) dy. \quad (98)$$

When comparing (98) with expression (57), section 5, we see that $\mathcal{Q}_\hbar(f)$ coincides with $\pi(\hat{f}_\hbar)$, where

$$\hat{f}_\hbar(x, y) = \int \frac{dp}{2\pi} e^{-ipy} f\left(x + \frac{\hbar}{2}y, p\right). \quad (99)$$

Here, \hat{f}_\hbar should be considered as an element of $\mathbb{R} > \triangleleft_{\alpha^\hbar} \mathbb{R}$. So, the quantum algebra coincides precisely with $C_r^*(\mathbb{R} > \triangleleft_{\alpha^\hbar} \mathbb{R})$, and the quantization map is given by $\mathcal{Q}_\hbar = \pi \circ \wedge_\hbar$, where $\wedge_\hbar(f) = \hat{f}_\hbar$.

In this perspective, the crucial step in the quantization process consists in the introduction of the algebra $C_r^*(\mathbb{R} > \triangleleft_{\alpha^\hbar} \mathbb{R})$, associated with the nontrivial action of the tangent vectors in configuration space.

Let us consider again the two dimensional torus T^2 . The identification $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ gives also a bijection between the set of continuous functions on the torus and the set of periodic continuous functions on \mathbb{R}^2 :

$$f(x + m, y + n) = f(x, y), \quad \forall m, n \in \mathbb{Z}.$$

The partial Fourier transform:

$$\mathcal{F} : f(x, y) \mapsto \tilde{f}(x, n) = \int_0^1 e^{-2\pi i n y} f(x, y) dy, \quad (100)$$

further allows us to pass from functions on the torus to functions on $S^1 \times \mathbb{Z}$, or periodic functions on $\mathbb{R} \times \mathbb{Z}$. For definiteness, let us consider as complete algebra of regular observables the subspace $\mathcal{A}(T^2)$ of those functions whose (total) Fourier transform possesses only a finite number of nonzero coefficients. It follows that the partial Fourier transform \mathcal{F} (100) is an isomorphism between $\mathcal{A}(T^2)$ and the space $\mathcal{A}(S^1 \times \mathbb{Z})$ of finite linear combinations of the functions F_{mk} in $S^1 \times \mathbb{Z}$ defined by:

$$F_{mk}(x, n) := e^{2\pi i m x} \delta_{nk}. \quad (101)$$

Let us consider the family of actions α^\hbar of \mathbb{Z} on S^1 , parametrized by real numbers $\hbar \in [0, 1[$ and defined by

$$\alpha_n^\hbar(x) = x + n\hbar \bmod 1, \quad n \in \mathbb{Z}. \quad (102)$$

With each of these actions, let us associate the semidirect product groupoid $S^1 > \triangleleft_{\alpha^\hbar} \mathbb{Z}$, as in examples 4 and 4a in section 5. Between two elements $g_1 = (x_1, n_1)$ and $g_2 = (x_2, n_2)$ of $S^1 > \triangleleft_{\alpha^\hbar} \mathbb{Z}$ such that $x_2 = \alpha_{n_1}^\hbar(x_1) = x_1 + n_1\hbar \bmod 1$, the composition rule is given by $g_1 g_2 = (x_1, n_1 + n_2)$. The inverse of the element $g = (x, n)$ is $g^{-1} = (\alpha_n^\hbar(x), -n) = (x + n\hbar \bmod 1, -n)$. Finally, let us note that both r -fibres and s -fibres can be identified with \mathbb{Z} . In particular, given an object $x_0 \in \text{Obj}(S^1 > \triangleleft_{\alpha^\hbar} \mathbb{Z}) \cong S^1$, the corresponding r -fibre is the set

$$G^{x_0} = \{(x_0, n), n \in \mathbb{Z}\}, \quad (103)$$

and the corresponding s -fibre is

$$G_{x_0} = \{(\alpha_n^\hbar(x_0), -n), n \in \mathbb{Z}\}. \quad (104)$$

The discrete structure of the fibres allows us to define the convolution algebra, which we now describe. Consider then the space $\mathcal{A}(S^1 \times \mathbb{Z})$, whose elements we identify with functions $F : \mathbb{R} \times \mathbb{Z} \rightarrow \mathbb{C}$ such that $F(x + m, n) = F(x, n)$, $m \in \mathbb{Z}$. The involution $*$ in $\mathcal{A}(S^1 \times \mathbb{Z})$ is defined by

$$F^*(g) = \bar{F}(g^{-1}), \text{ or} \quad (105)$$

$$F^*(x, n) = \bar{F}(x + n\hbar, -n), \quad F \in \mathcal{A}(S^1 \times \mathbb{Z}). \quad (106)$$

In particular for the basis elements F_{mk} (101) we get

$$F_{mk}^* = e^{2\pi i m k \hbar} F_{-m-k}. \quad (107)$$

The convolution \star is defined by

$$(F \star G)(x, n) = \sum_{m \in \mathbb{Z}} F(x, m) G(x + m\hbar, n - m). \quad (108)$$

$\mathcal{A}(S^1 \times \mathbb{Z})$ is therefore an involutive algebra with identity, namely the function $F_{00}(x, n) = \delta_{n0}$. In particular, the convolution (108) of basis elements leads to

$$F_{mk} \star F_{m'k'} = e^{2\pi i m' k \hbar} F_{m+m' \ k+k'}. \quad (109)$$

One can easily show that this algebra is generated by the two elements F_{10} e F_{01} . Furthermore, since F_{10} are F_{01} are unitary and satisfy the commutation relations

$$F_{01} \star F_{10} = e^{2\pi i \hbar} F_{10} \star F_{01}, \quad (110)$$

we conclude that the algebra in question is none other than the well know *universal rotation algebra* $\mathcal{A}_\hbar^{\text{rot}}$, parametrized by \hbar , which is precisely defined as the $*$ -algebra generated by two elements u and v subject to the relations $u^*u = uu^* = v^*v = vv^* = 1$ and $vu = e^{2\pi i \hbar} uv$ [29]. The *rotation C^* -algebra* A_\hbar^{rot} is by definition the completion of $\mathcal{A}_\hbar^{\text{rot}}$ with respect to the norm

$$\|a\| := \sup\{\|\pi a\| : \pi \text{ is a representation of } \mathcal{A}_\hbar^{\text{rot}}\}, \quad a \in \mathcal{A}_\hbar^{\text{rot}}, \quad (111)$$

and satisfies the folowing universality property [29]

Theorem 1 *Let A be a C^* -algebra with two elements u', v' satisfying the same relations as the generators u, v of A_\hbar^{rot} . Then there exists a morphism $\varphi : A_\hbar^{\text{rot}} \rightarrow A$ such that $v \mapsto v'$ and $u \mapsto u'$. If \hbar is irrational then φ is an isomorphism between A_\hbar^{rot} and the smallest closed subalgebra of A that contains u and v .*

This result shows immediately that $C_r^*(S^1 > \triangleleft_{\alpha\hbar} \mathbb{Z})$ is isomorphic to A_\hbar^{rot} when \hbar is irrational. In the \hbar rational case, and for an arbitrary algebra A , the map φ of the above theorem is not necessarily injective. In the present case, however, injectivity is clearly ensured, and therefore $C_r^*(S^1 > \triangleleft_{\alpha\hbar} \mathbb{Z})$ is isomorphic to A_\hbar^{rot} , $\forall \hbar$.

The structure of the rotation algebras A_\hbar^{rot} depends heavily on the value of \hbar : for $\hbar = 0$ we recover, as expected, the algebra $C(T^2)$ of continuous functions on the torus; the irrational \hbar case is extremely interesting from the point of view of noncommutative geometry and has been extensively studied [27, 10, 7, 30]. Let us focus on the rational (nonzero) \hbar case, following [7, 30]. As we will see shortly, the quantization (96) described in the previous section will emerge here quite naturally.

Let us first note that two distinct values \hbar, \hbar' such that $\hbar + \hbar' = 1$ lead to the same algebra, i.e. $A_{1-\hbar}^{\text{rot}}$ is isomorphic to A_{\hbar}^{rot} . Let then $\hbar = \frac{K}{N}$, with $K, N \in \mathbb{N}$ and $K \leq N/2$. Taking into account the convolution (109), one can easily check that the elements of the form $F_{mN \ kN}$, $m, k \in \mathbb{Z}$, commute with the generators, and therefore belong to the centre of the algebra. So, given any irreducible representation π^0 , the image $\pi^0(F_{mN \ kN})$ of those elements must be proportional to the identity. It follows that the irreducible representations of the algebra A_{\hbar}^{rot} , with $\hbar = K/N$ as above, are finite dimensional, of dimension N . Note also that, being finite dimensional, the irreducible representation is unique (modulo unitary equivalence).

A convenient irreducible representation can be easily found, as follows. Let $\mathcal{H}_N \cong \mathbb{C}^N$ be the Hilbert space generated by an orthonormal set of N vectors, say $\{v^0, v^1, \dots, v^{N-1}\}$. In $B(\mathcal{H}_N)$ consider unitary operators $\mathcal{U}_N(K)$, $\mathcal{V}_N(K)$ such that

$$\mathcal{U}_N(K)v^k = e^{2\pi i k/N} v^k, \quad (112)$$

$$\mathcal{V}_N(K)v^k = v^{k-K \bmod N}. \quad (113)$$

We obtain immediately the commutation relations:

$$\mathcal{V}_N(K)\mathcal{U}_N(K) = e^{2\pi i K/N} \mathcal{U}_N(K)\mathcal{V}_N(K). \quad (114)$$

The pair $\mathcal{U}_N(K)$, $\mathcal{V}_N(K)$ therefore satisfies the relations (110) corresponding to $\hbar = K/N$, which shows that the $*$ -morphism $\pi_{K,N} : A_{\hbar}^{\text{rot}} \rightarrow B(\mathcal{H}_N)$ given by

$$\pi_{K,N}(F_{10}) = \mathcal{U}_N(K) \quad (115)$$

$$\pi_{K,N}(F_{01}) = \mathcal{V}_N(K) \quad (116)$$

is well defined and is a representation, obviously irreducible, of $A_{K/N}^{\text{rot}}$.

Let us finally construct a family $\mathcal{Q}_{K/N}$ of quantizations of the 2-torus. As already suggested, let us adopt as complete algebra of regular observables the subalgebra $\mathcal{A}(T^2) \subset C(T^2)$ of those functions whose Fourier transform possesses only a finite number of nonzero coefficients. Let then $\wedge_{K/N} : \mathcal{A}(T^2) \rightarrow C_r^*(S^1 \times \triangleleft_{\alpha \hbar} \mathbb{Z})$ denote the maps given by

$$f(x, y) \mapsto \hat{f}_{K/N}(x, n) = \int_0^1 e^{-2\pi i n y} f(x + nK/2N, y) dy, \quad (117)$$

which correspond to (99). For the elements $e^{2\pi i(mx+ky)}$ of the base we get simply

$$e^{2\pi i(mx+ky)} \xrightarrow{\wedge_{K/N}} e^{\pi i m k K/N} F_{mk}. \quad (118)$$

The quantizations maps are then $\mathcal{Q}_{K/N} = \pi_{K,N} \circ \wedge_{K/N}$, leading to

$$\mathcal{Q}_{K/N}(e^{2\pi i(mx+ky)}) = e^{\pi i m k K/N} \mathcal{U}_N(K)^m \mathcal{V}_N(K)^k \in B(\mathcal{H}_N). \quad (119)$$

In particular for $K = 1$, this coincides with the quantization put forward in the previous section, expressed namely in quantization rules (92), (94) and (96).

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